AD-787 041

LOW-FREQUENCY SOUND PROPAGATION IN A FLUCTUATING INFINITE OCEAN

C. G. Callan, et al

Stanford Research Institute

Prepared for:

Defense Advanced Research Projects

April 1974

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

Technical Report JSR-73-10 April 1974

LOW-FREQUENCY SOUND PROPAGATION IN A FLUCTUATING INFINITE OCEAN

By: C. G. CALLAN F. ZACHARIASEN

Contract No. DAHC15-73-C-0370
ARPA Order No. 2504
Program Code No. 3K10
Date of Contract: 2 April 1973
Contract Expiration Date: 31 December 1974
Amount of Contract: $697,623

Approved for public release; distribution unlimited.

Sponsored by

DEFENSE ADVANCED RESEARCH PROJECTS AGENCY
ARPA ORDER NO. 2504

JASON

STANFORD RESEARCH INSTITUTE
Menlo Park, California 94025 · U.S.A.

Copy No. 18

iB

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD - 787041

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

LOW-FREQUENCY SOUND PROPAGATION IN A FLUCTUATING INFINITE OCEAN

5. TYPE OF REPORT & PERIOD COVERED

Technical Report JSR-73-10

6. PERFORMING ORG. REPORT NUMBER

SRI Project 3000

7. AUTHOR(s)

C. G. Callan F. Zachariasen

8. CONTRACT OR GRANT NUMBER(s)

Contract DAHC15-73-C-0370

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Stanford Research Institute
Menlo Park, California 94025

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Defense Advanced Research Projects Agency
1400 Wilson Boulevard
Arlington, Virginia 22209

12. REPORT DATE

April 1974

13. NO. OF PAGES

51

14. MONITORING AGENCY NAME & ADDRESS (if diff. from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

Approved for Public Release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

LF sound pressure
Long distance propagation
Fluctuating ocean
Analytic approximation
Extend range of validity
Temporal and spatial fluctuations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An approximation is described permitting a relatively simple analytic expression for various statistical averages of the sound-pressure field far from a source in a weakly fluctuating medium. In the low-frequency regime this approximation will permit valid calculations to larger distances than the conventional eikonal, or geometrical optics on straight paths, or perturbation-theory formulae, provided that $(kL)^2(\delta c/c)^{4/3} < 1$, where (continued)

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19. KEY WORDS (Continued)

20 ABSTRACT (Continued)

$\delta c/c$ and L are the fractional sound-speed fluctuations and correlation length respectively, and $2\pi/k$ is the sound wavelength. The source-to-receiver range over which the method is valid is $R = L(1/kL)^2(c/\delta c)^2$. The method is used to estimate mean square fluctuations of the sound pressure and coherence lengths in an isotropic infinite ocean, and it is pointed out that its use in the more realistic situation of cylindrical symmetry is not appreciably more difficult. More careful calculations of these quantities are presented, as an illustration of the use of the method, for the example when the correlation function of the sound-speed fluctuations is exponential.

ABSTRACT

An approximation is described permitting a relatively simple analytic expression for various statistical averages of the sound-pressure field far from a source in a weakly fluctuating medium. In the low-frequency regime this approximation will permit valid calculations to larger distances than the conventional eikonal, or geometrical optics on straight paths, or perturbation-theory formulae, provided that $(kL)^2(\delta c/c)^{4/3} < 1$, where $\delta c/c$ and L are the fractional sound-speed fluctuations and correlation length respectively, and $2\pi/k$ is the sound wavelength. The source-to-receiver range over which the method is valid is $R = L(1/kL)^2(c/\delta c)^2$. The method is used to estimate mean square fluctuations of the sound pressure and coherence lengths in an isotropic infinite ocean, and it is pointed out that its use in the more realistic situation of cylindrical symmetry is not appreciably more difficult. More careful calculations of these quantities are presented, as an illustration of the use of the method, for the example when the correlation function of the sound-speed fluctuations is exponential.

CONTENTS

ILLUSTRATIONS

I STATEMENT OF THE PROBLEM

This report is motivated by the desire to understand the statistical fluctuations observed in the long-distance propagation of low-frequency sound in the deep ocean. There are temporal and spatial fluctuations of the sound speed in the ocean, caused by, among other things, the presence of internal waves, and these lead to corresponding fluctuations in the received signal from a sound source. Suppose a series of pulses is emitted from a sound source and picked up by a distant receiver. During the very short travel time of a single pulse, the sound speed in the ocean is essentially fixed, but between successive pulses it can change. Thus the variations in the sequence of received signals probe the variations in the sound speed. Measurements have been made under varying conditions of the statistical properties of such sequences of received signals,[1]* and there have also been speculations on the statistical properties of the fluctuations in sound speed.[2] The question is whether or not these properties can be connected theoretically.

From the above physical problem we abstract the mathematical problem that will be discussed in this report, as follows: what are the statistics of sound propagation in an infinite gaussian random medium? Suppose we have a CW point source of sound of wavelength λ located at the origin in an infinite medium with sound speed $c + \delta c(\vec{x})$, where c is the mean sound speed and where the spatial variations $\delta c(\vec{x})$ are small: $\delta c(\vec{x}) \ll c$. Suppose $\delta c(\vec{x})$ is given, or more precisely, suppose the correlation function

---

*References are listed at the end of the report.

$$C(\vec{x}_1 - \vec{x}_2) \equiv 4\left(\frac{2\pi}{\lambda}\right)^4 \left\langle \frac{\delta c(\vec{x})_1}{c} \frac{\delta c(\vec{x})_2}{c} \right\rangle \tag{1.1}$$

is given. We wish to calculate the pressure $p(\vec{x})$ at an arbitrary point $\vec{x}$ in the medium; or more precisely, we wish to calculate the moments $\langle p(\vec{x})\rangle$, $\langle p(\vec{x}_1)p(\vec{x}_2)\rangle$ and $\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle$ at arbitrary points $\vec{x}_1$ and $\vec{x}_2$. In particular, we may wish to know $\langle |p(\vec{x})|^2\rangle$, the received mean square pressure at a point; the pressure coherence distance D such that when $|\vec{x}_1 - \vec{x}_2| > D$, $\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle$ vanishes; the phase of the received pressure, and so forth.

When $\delta c(\vec{x})/c$ is very small, the sound pressure satisfies the wave equation

$$(\nabla^2 + k^2)p(\vec{x}) = V(\vec{x})p(\vec{x}) \tag{1.2}$$

where

$$V(x) \equiv 2k^2 \delta c(\vec{x})/c \tag{1.3}$$

and where $k = 2\pi/\lambda$ is the wavenumber of the emitted sound. For a point source of strength S, the pressure must satisfy the boundary condition that $(\nabla^2 + k^2)p(\vec{x}) \rightarrow S\,\delta^3(x)$ as $\vec{x} \rightarrow 0$; we shall usually choose $S = 1$ so that we are dealing with a source of unit strength.

The parameters that will be relevant to our discussion are λ, or k; $R = |\vec{x}|$, the distance from source to receiver; and L, the sound-speed correlation length such that $C(\vec{x}_1 - \vec{x}_2)$ vanishes if $|\vec{x}_1 - \vec{x}_2| \gtrsim L$. For the purposes of explicit illustration, we shall sometimes use the example $C(x) = C\, e^{-|\vec{x}|/L}$, with $C = 4k^4(\delta c/c)^2$.

Typical numerical values of these parameters are: $\lambda \sim 10$ m, $L \sim 100$ m, $R \sim$ a few tens to a few thousand km, and $\delta c/c \sim 10^{-4}$. These values are not out of line with what one might expect for the oceanic sound-propagation problem that motivates us.

At first glance, since the wavelength is small, one might be tempted to think that geometrical optics provides a viable approximation with which to calculate the sound-pressure field. However, in order to obtain explicit answers, one must use geometrical optics on straight-line propagation paths, and, as explained in Appendix A, the condition for the validity of this approximation is $R \lesssim L(c/\delta c)^{2/3}$, which, with our typical parameter values, requires $R \lesssim 50$ km. This is too short a propagation distance to satisfy us, so we are compelled to find a better approximation. How this is done is the subject of this report.

II STATEMENT OF CONCLUSIONS

To spare the uninterested reader the pain of paging through a lot of arithmetic, we shall present our conclusions here:

(1) An excellent approximation to the average received pressure is

$$\langle p(x) \rangle = \frac{1}{4\pi R} e^{i[k - \Sigma(k)/2k]R} \tag{2.1}$$

where

$$\Sigma(k) = \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{C}(\vec{k} - \vec{q})}{k^2 - q^2 + i\epsilon} \tag{2.2}$$

and where $\tilde{C}$ is the Fourier transform of the sound-speed correlation function:

$$\tilde{C}(\vec{q}) = \int d^3 \vec{x}\, e^{-i\vec{q}\cdot\vec{x}}\, C(\vec{x}) \quad . \tag{2.3}$$

This expression is valid provided $(kL)^3(\delta c/c)^2 << 1$, independently of the distance from source to receiver.

Since $\Sigma(k)$ is complex, $\langle p(\vec{x}) \rangle$ is exponentially damped in distance. We have

$$|\langle p(\vec{x}) \rangle| = e^{-\alpha R}/4\pi R \tag{2.4}$$

where, for the case of spherical symmetry, where $\tilde{C}(\vec{q}) = \tilde{C}(q^2)$,

$$\alpha = \frac{1}{16\pi} \int_{-1}^{1} d(\cos\theta) \tilde{C}[2k^2(1 - \cos\theta)] \quad . \tag{2.5}$$

In particular, for the exponential example where $C(\vec{x}) = C\, e^{-R/L}$, we find (when $kL >> 1$)

$$\alpha = CL/4k^2 \quad . \tag{2.6}$$

Thus, if $C = 4k^4(\delta c/c)^2$, we get

$$\alpha = k^2 L \; (\delta c/c)^2 \tag{2.7}$$

so that the damping distance is

$$\frac{1}{\alpha} = \left(\frac{\lambda}{2\pi}\right)\left(\frac{1}{kL}\right)\left(\frac{c}{\delta c}\right)^2 \quad . \tag{2.8}$$

With our typical values for the parameters, this distance is 3,000 km.

(2) As outlined in Appendix A, geometrical optics with straight-line paths is a valid approximation for distances $R < R_0$, where $R_0 = \min[L(c/\delta c)^{2/3},\ L(kL)^{4/3}]$. Provided that $R_1/R_0 = (1/kL)^2 (c/\delta c)^{4/3} > 1$, an improvement over geometrical optics valid out to greater distances $R < R_1$, where

$$R_1 = L\left(\frac{1}{kL}\right)^2\left(\frac{c}{\delta c}\right)^2 \quad , \tag{2.9}$$

is the supereikonal approximation, in which the pressure is given by

$$p(\vec{x}) = i\int \frac{d^3\vec{q}}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}} \int_0^\infty d\xi\, e^{-i\xi[q^2 - k^2 - I(\xi,x) - i\epsilon]} \tag{2.10}$$

where

$$I(\xi,\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3}\, \tilde{V}(q) \int_0^1 ds\, e^{-is\vec{q}\cdot\vec{x}}\, e^{-i\xi s(1-s)q^2} \tag{2.11}$$

and where $\tilde{V}(q)$ is the Fourier transform of $V(x)$:

$$\tilde{V}(\vec{q}) = \int d^3\vec{x}\, e^{-i\vec{q}\cdot\vec{x}}\, V(\vec{x}) \quad . \tag{2.12}$$

For our typical parameters, the conditions of validity permit the use of this approximation out to a distance $R \sim 3000$ km.

Note that this range of validity is the same as the damping distance α^{-1}.

(3) While this is a result in closed form, and one that can therefore be used for numerical purposes, it is convenient to make a further approximation in order to obtain a more explicit expression. Under the condition that

$$kL \gg 1 \tag{2.13}$$

a valid approximation to Eqs. (2.10) and (2.11) is

$$p(\vec{x}) = \langle p(\vec{x}) \rangle \, e^{X(\vec{x}) - 1/2 \langle X^2(\vec{x}) \rangle} \tag{2.14}$$

where

$$X(x) = \frac{1}{4\pi} \int d^3\vec{y} \, \frac{|\vec{x}|}{|\vec{x}-\vec{y}|\,|\vec{y}|} \, e^{ik(|\vec{x}-\vec{y}| + |\vec{y}| - |\vec{x}|)} V(\vec{y}) \; . \tag{2.15}$$

(4) From Conclusions (3) above, one can obtain expressions for the statistical averages of interest. First,

$$\frac{\langle p(\vec{x}_1) p(\vec{x}_2) \rangle}{\langle p(\vec{x}_1) \rangle \langle p(\vec{x}_2) \rangle} = e^{[\tilde{C}(0)/16k]} \, e^{[i\tilde{C}(0)/8\pi k](C + \log 4kR)} \tag{2.16}$$

where C = 0.577... is Euler's constant. Note this is independent of $\vec{x}_1 - \vec{x}_2$; there is infinite coherence length for this quantity.

Second, for the case of spherical symmetry, the mean square pressure is

$$\frac{\langle |p(x)|^2 \rangle}{|\langle p(x) \rangle|^2} = \exp \frac{R}{4\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \left(\sin^{-1} \sqrt{z}\right) \tilde{C}(2k^2 z) \tag{2.17}$$

where

$$\tilde{C}(q^2) = \int d^3\vec{x}\, e^{-i\vec{q}\cdot\vec{x}}\, C(\vec{x}) \quad . \tag{2.18}$$

To illustrate Eqs. (2.16) and (2.17), let us use our exponential model

$$C(x) = C\, e^{-|\vec{x}|/L} \quad . \tag{2.19}$$

Then, Eq. (2.16) becomes

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = e^{i(CL^3/k)(C+\log 4kR-i\pi/2)} \tag{2.20}$$

while Eq. (2.17) becomes

$$\frac{\langle |p(\vec{x})|^2\rangle}{|\langle p(\vec{x})\rangle|^2} = \exp \frac{RCL}{k^2} \quad . \tag{2.21}$$

If we set $C = 4k^4(\delta c/c)^2$, these expressions are replaced by

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = e^{i\cdot 4(kL)^3(\delta c/c)^2(C+\log 4kR-i\pi/2)} \tag{2.22}$$

and

$$\frac{\langle |p(\vec{x})|^2\rangle}{|\langle p(\vec{x})\rangle|^2} = e^{4(kL)(\delta c/c)^2(kR)} \quad . \tag{2.23}$$

One must keep in mind the conditions of validity for these expressions--namely, $R \ll L(1/kL)^2(c/\delta c)^2$. Thus, the exponent in Eq. (2.23) cannot become as large as 4. Beyond this value of R the exponential growth with distance ceases, and damping as described in Conclusion (1), above, sets in. As a result, Eq. (2.23) cannot be believed at distances beyond those at which the exponent is of order one.

(5) Finally, while the remaining quantities we set out to compute are more difficult to obtain explicitly, rough dimensional estimates of them can be made.

For spherical symmetry,

$$\frac{\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle^*} \sim e^{[\tilde{C}(0)/(kL)^2][1-(x_\perp/L)^2]R} \tag{2.24}$$

where $\vec{x} = \vec{x}_1 - \vec{x}_2$, and $x_\perp$ is the component of $\vec{x}$ perpendicular to $\vec{R} = (\vec{x}_1 + \vec{x}_2)/2$. For cylindrical symmetry, if $\vec{R}$ lies in a "horizontal" plane transverse to the symmetry axis, the rough estimate, Eq. (2.24), is changed to

$$\frac{\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle^*} \sim \exp \frac{\tilde{C}(0)}{k^2 L_H L_V}\left[1 - \left(\frac{x_V}{L_V}\right)^2 - \left(\frac{x_H}{L_H}\right)^2\right] \tag{2.25}$$

where L_H and L_V denote the "horizontal" and "vertical" (transverse and parallel to the symmetry axis, which we call "vertical") sound pressure correlation lengths, and x_H and x_V are the corresponding components of x.

If we estimate $\tilde{C}(0) \sim k^4(\delta c/c)^2 L^3$ and $\tilde{C}(0) \sim k^4(\delta c/c)^2 L_H^2 L_V$ for the spherical and cylindrical cases respectively, we find for the cutoff distances (i.e., the pressure coherence distances):

$$D_\perp \sim \lambda\sqrt{L/R}\,(c/\delta c)$$

$$D_H \sim \lambda\sqrt{L_H/R}\,(c/\delta c)(L_H/L_V)^{1/4}$$

$$D_V \sim \lambda\sqrt{L_H/R}\,(c/\delta c)(L_V/L_H)^{3/4} \quad .$$

With our standard parameters, all of these are of order 10^3 m when R is 100 km.

III THE DERIVATION OF THE APPROXIMATION

A. The Perturbation Series

We shall now show how to derive our improved expression for sound propagation in a random medium. The fundamental equation for the propagation of sound of frequency ω is

$$(\nabla^2 + k^2)p(\vec{x}) = V(\vec{x})p(\vec{x}) \tag{3.1}$$

where $k = \omega/c$, c is the mean sound velocity, $V(\vec{x}) = 2k^2[\delta c(\vec{x})/c]$, $\delta c(\vec{x})$ is the (small) difference between the local sound velocity and c, and $p(x)$ is the sound pressure field.

If $V = 0$, the solution of Eq. (3.1) is

$$P_0(\vec{x}) = \frac{e^{ik|\vec{x}|}}{4\pi\,|\vec{x}|} \tag{3.2}$$

for a point source of unit strength at $\vec{x} = 0$. Since V is small, it makes sense to study an expansion of the solution in powers of V. To that end, we introduce the Greens function, $G_0(\vec{x} - \vec{y})$, satisfying

$$(\nabla^2 + k^2)G_0(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})$$

and note that the solution of Eq. (3.1) may be written as

$$p(\vec{x}) = p_0(\vec{x}) + \int d^3\vec{x}^{\,\prime}\; G_0(\vec{x} - \vec{x}^{\,\prime})\; V(\vec{x})p(\vec{x}^{\,\prime}) \; . \tag{3.3}$$

Iterating yields the desired expansion in V:

$$\begin{aligned} p(\vec{x}) = p_0(\vec{x}) &+ \int d^3x_1\; G_0(\vec{x} - \vec{x}_1)V(\vec{x}_1)p_0(\vec{x}_1) \\ &+ \int d^3x_1 d^3x_2\; G_0(\vec{x} - \vec{x}_1)V(\vec{x}_1)\; G_0(x_1 - \vec{x}_2)V(\vec{x}_2)p_0(\vec{x}_2) \\ &+ \dots \; . \end{aligned} \tag{3.4}$$

To facilitate further manipulation we pass to momentum space, making use of the Fourier transforms

$$\tilde{G}_0(q) = \int dx\, e^{-iqx}\, G_0(x) = \frac{1}{k^2 - q^2}$$

$$\tilde{p}_0(q) = \int dx\, e^{-iqx}\, p_0(x) = \frac{1}{k^2 - q^2} \quad . \tag{3.5}$$

We obtain

$$\tilde{p}(\vec{q}) = \sum_n \tilde{p}_n(\vec{q}) \quad ;$$

$$\tilde{p}_n(\vec{q}) = \int \prod_{i=1}^{n} d^3\vec{q}_i V(\vec{q}_i) \frac{1}{(q^2-k^2)[(\vec{q}+\vec{q}_1)^2-k^2]\ldots[(q+\sum^n q_i)^2-k^2]}$$

$$= (n!) \int \prod^n d^3\vec{q}_i V(\vec{q}_i) \frac{\prod^n d\alpha_i \;\delta(1-\sum^n \alpha_i)}{\left\{\alpha_0(q^2-k^2)+\alpha_1[(\vec{q}+\vec{q}_1)^2-k^2]+\ldots+\alpha_n[(\vec{q}+\sum^n q_1)^2-k^2]\right\}^{n+1}} . \tag{3.6}$$

The last step makes use of the identity

$$\frac{1}{a_1 \ldots a_n} = (n-1)! \int \prod_{i=1}^{n} d\alpha_i \frac{\delta(1-\sum^n \alpha_i)}{\left[\sum^n \alpha_i a_i\right]^n} \tag{3.7}$$

The denominator in Eq. (3.6) may be rearranged as

$$D = (\vec{p} + \sum^n s_i \vec{q}_i)^2 - k^2 + \sum_{i=1}^{n} s_i(1-s_i) q_i^2 + X \tag{3.8}$$

where

$$X = 2 \sum_{i>j} s_i (1-s_j) \vec{q}_i \cdot \vec{q}_j$$

and

$$s_i = \sum_{j=i}^{n} \alpha_j \quad . \tag{3.9}$$

We can now replace the integrations over the α_i in Eq. (3.6) by integrations over the s_i. We have only to remember that the s_i range between 0 and 1 and are ordered $s_{i+1} < s_i$. Then

$$\tilde{p}_n(q) = n! \int \frac{\prod^n d^3\vec{q}_i V(\vec{q}_i) \prod_{i=1}^n ds_i}{\left[(\vec{q}+\sum^n s_i\vec{q}_i)^2 - k^2 + \sum^n s_i(1-s_i)\vec{q}_i^{\,2} + X\right]^{n+1}} \tag{3.10}$$

and therefore

$$p_n(x) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \tilde{p}_n(q)$$

$$= n! \int d^3q\, e^{i\vec{q}\cdot\vec{x}} \int \frac{\prod^n d^3\vec{q}_i V(\vec{q}_i) e^{-is_i\vec{q}_i\cdot\vec{x}} \prod^n ds_i}{\left[q^2 - k^2 + \sum s_i(1-s_i)\, q_i^2 + X\right]^{n+1}} \quad . \tag{3.11}$$

So far we have made no approximations and it is of course impossible to sum the series exactly. We shall therefore neglect the quantity X and show that with this simplification the series becomes summable. Later on we shall discuss the conditions under which this approximation is valid.

Having dropped X, we see that the integrand of Eq. (3.11) is symmetric in all the s_i so that in the integration over s_i we may drop the restriction $s_i > s_{i+1}$ and absorb the factor of n! All s_i then run independently from 0 to 1 and we have

$$p(x) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \int \prod_{i=1}^n \int ds \frac{\frac{d^3\vec{q}_i}{(2\pi)^3} V(\vec{q}_i)\, e^{-is_i\vec{q}_i\cdot\vec{x}}}{\left[q^2 - k^2 + \sum^n s_i(1-s_i) q_i^2\right]^{n+1}}$$

$$= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \frac{1}{n!}\left(\frac{\partial}{\partial k^2}\right)^n \int \frac{\prod_{i=1}^n \int ds_i \frac{d^3\vec{q}_i}{(2\pi)^3} V(\vec{q}_i)\, e^{-is_i\vec{q}_i\cdot\vec{x}}}{\left[q^2 - k^2 + \sum^n s_i(1-s_i) q_i^2\right]}$$

$$p(x) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \frac{1}{n!} \left(\frac{\partial}{\partial k^2}\right) i \int_0^\infty d\beta \, e^{-i\beta[q^2-k^2+\sum^n s_i(1-s_i)q_i^2 - i\epsilon]}$$

$$\prod_{i=1}^n ds_i \frac{d^3\vec{q}}{(2\pi)^3} e^{-is_i\vec{q}_i\cdot\vec{x}}$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} \, i \int_0^\infty d\beta \, e^{-i\beta(q^2-k^2-i\epsilon)} \sum \frac{1}{n!} (i\beta)^n$$

$$\left[\int_0^1 ds \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) \, e^{-is\vec{q}\cdot\vec{x}-i\beta s(1-s)q^2}\right]^n$$

so that finally we may write

$$p(\vec{x}) = i \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \int_0^\infty d\beta \, e^{-i\beta(q^2-k^2-I(\beta,\vec{x})-i\epsilon)} \tag{3.12}$$

with

$$I(\beta,\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) \int_0^1 ds \, e^{-is\vec{q}\cdot\vec{x}-i\beta s(1-s)q^2} \quad . \tag{3.13}$$

This expression may appear somewhat unwieldy, but it is at least a closed form, and we shall shortly see that it can be considerably simplified.

At this point we note that if we had dropped all the terms of order q^2 in Eq. (3.11) we would have obtained, instead of Eq. (3.13), the result

$$I(\beta,\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \int_0^1 ds \, e^{-is\vec{q}\cdot\vec{x}} \, V(\vec{q})$$

$$= \int_0^1 ds \, V(s\vec{x}) \tag{3.14}$$

and hence, instead of Eq. (3.12)

$$p(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2-k^2-\int_0^1 ds\ V(sx)}$$

$$= \frac{1}{4\pi|\vec{x}|} \exp i|\vec{x}|\sqrt{k^2+\int_0^1 ds\ V(s\vec{x})} \quad . \tag{3.15}$$

This expression is just the usual eikonal approximation answer for p(x), and we expect that our new answer, Eq. (3.12), which we shall call the supereikonal approximation, represents an improvement over this. The next question to discuss is, when is this answer an improvement and by how much?

B. Error Estimates

Our concern here is not to show precisely what are the limits of validity of our expressions, but simply to argue heuristically that they should hold over a much wider range than the usual eikonal result, at least under a wide range of conditions of interest. To this end we return to the exact expression for $p_n(\vec{x})$, Eq. (3.11), and expand the denominator in powers of the terms quadratic in the $\vec{q}_i$, keeping only the zeroth and first-order terms:

$$p(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3} V(\vec{q}_i)\, e^{-is_i\vec{q}_i\cdot\vec{x}}\, ds_i$$

$$\left\{ \frac{n!}{(q^2-k^2)^{n+1}} - (n+1)!\, \frac{\sum^n s_i(1-s_i)q_i^2}{(q^2-k^2)^{n+2}} - \frac{(n+1)!X}{(q^2-k^2)^{n+2}} + \ldots \right\} \quad . \tag{3.16}$$

Here we must remember that the s_i range from 0 to 1 and are ordered $s_i < s_{i+1}$. We shall call the three contributions to $p(\vec{x})$ corresponding to the three terms in curly brackets $p_0(\vec{x})$, $p_1(\vec{x})$, and $p_2(\vec{x})$:

$$P_0(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \frac{n!}{(q^2-k^2)^{n+1}} \Pi ds_i \frac{d^3\vec{q}_i}{(2\pi)^3} e^{-is_i\vec{q}_i\cdot\vec{x}} V(\vec{q}_i)$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{\left[q^2-k^2-\int_0^1 ds\, V(s\vec{x})\right]}$$

$$= \frac{1}{4\pi|\vec{x}|} \exp i|\vec{x}| \sqrt{k^2 + \int_0^1 ds\, V(s\vec{x})} \quad ; \tag{3.17}$$

$$p_1(\vec{x}) = -\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum_n \frac{(n+1)!}{(q^2-k^2)^{n+2}} \Pi ds_i \frac{d^3\vec{q}_i}{(2\pi)^3} e^{-is_i\vec{q}_i\cdot\vec{x}} V(\vec{q}_i) \sum^n s_i(1-s_i) q_i^2$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{\left[q^2-k^2-\int_0^1 ds V(s\vec{x})\right]^3} \int_0^1 ds \frac{(1-s)}{s} \nabla^2 V(s\vec{x})$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial k^2}\right)^2 \frac{\exp i|\vec{x}| \sqrt{k^2 + \int_0^1 ds\, V(s\vec{x})}}{4\pi|\vec{x}|} \int_0^1 ds \frac{(1-s)}{s} \nabla^2 V(s\vec{x}) \; ; \tag{3.18}$$

$$p_2(x) = -2\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \sum \frac{(n+1)!}{(q^2-k^2)^{n+2}} \overset{n}{\Pi} ds_i \frac{d^3\vec{q}_i}{(2\pi)^3} e^{-is_i\vec{q}_i\cdot\vec{x}} V(\vec{q}_i)$$

$$\sum_{i>j}^{n} s_i(1-s_j)\vec{q}_i\cdot\vec{q}_j$$

$$= 2\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{\left[q^2-k^2+\int_0^1 ds\, V(s\vec{x})\right]^4} \int_0^1 ds_1 \int_0^{s_1} ds_2 \frac{(1-s_1)}{s_1}$$

$$\vec{\nabla}V(s_1\vec{x})\cdot\vec{\nabla}V(s_2\vec{x})$$

$$= +\frac{1}{3}\left(\frac{\partial}{\partial k^2}\right)^3 \frac{\exp i|\vec{x}| \sqrt{k^2 + \int_0^1 ds\, V(s\vec{x})}}{4\pi|\vec{x}|} \int_0^1 ds_1 \int_0^{s_1} ds_2 \frac{(1-s_1)}{s_1}$$

$$\vec{\nabla}V(s_1\vec{x})\cdot\vec{\nabla}V(s_2\vec{x}) \; . \tag{3.19}$$

The manipulations that allow us to express p_1 and p_2 in these relatively simple forms are straightforward but tedious and will not be detailed.

Now p_0 is just the normal eikonal expression for the pressure field, p_1 is a correction to this of the type that we have kept in the supereikonal approximation, and p_2 is a correction we have thrown away. Therefore, the criterion for the normal eikonal to be valid is that $p_1/p_0 < 1$, and the criterion for the supereikonal to be valid is that $p_2/p_1 < 1$.

In the applications we shall consider, V is sufficiently small that we may set $\sqrt{k^2+\int_0^1 ds\, V(s\vec{x})}$ equal to k. Also, we are interested in the behavior of the various p_i only for large $R = |\vec{x}|$. Thus the indicated operations $\partial/\partial k^2$ may be replaced by multiplication by R/2k as far as the leading asymptotic behavior goes. This leaves us with

$$\frac{p_1}{p_0} \underset{R\to\infty}{\longrightarrow} \frac{R^2}{8k^2}\int_0^1 ds\, \frac{(1-s)}{s}\, \nabla^2 V(s\vec{x}) \equiv \frac{R^2}{8k^2}\,\Phi \; ; \tag{3.20}$$

$$\frac{p_2}{p_1} \underset{R\to\infty}{\longrightarrow} \frac{1}{3}\,\frac{R}{k}\, \frac{\displaystyle\int_0^1 ds_1 \int_0^{s_1} ds_2\, \frac{(1-s_1)}{s_1}\, \vec{\nabla}V(s_1\vec{x})\cdot\vec{\nabla}V(s_2\vec{x})}{\displaystyle\int_0^1 ds\, \frac{(1-s)}{s}\, \nabla^2 V(sx)} \equiv \frac{R}{3k}\,\Omega \; . \tag{3.21}$$

To discuss the behavior of these ratios as a function of R we need to know the expectation of Ω^2 and Φ^2. Since the potential, $V(\vec{x})$, is assumed to be a gaussian random variable satisfying $\langle V(\vec{x})V(\vec{y})\rangle = C(\vec{x}-\vec{y})$, we have

$$\langle \Phi^2\rangle = \int_0^1 ds_1 \int_0^1 ds_2\, \frac{(1-s_1)}{s_1}\,\frac{(1-s_2)}{s_2}\, s_1^2\, s_2^2\, \left[\nabla^4 C(\vec{X})\right]_{\vec{X}=(s_1-s_2)\vec{x}} \tag{3.22}$$

and

$$\langle \Omega^2 \rangle = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^1 ds_1' \int_0^{s_1'} ds_2' \frac{(1-s_1)}{s_1} \frac{(1-s_2')}{s_2'} s_1 s_2 s_1' s_2'$$

$$\left\{ \nabla^2 C(\vec{\rho}) \nabla^2 C(\vec{\sigma}) \Big|_{\substack{\vec{\rho} = (s_1 - s_2)x \\ \vec{\sigma} = (s_1' - s_2')x}} \right.$$

$$+ \nabla_i \nabla_j C(\vec{\rho}) \nabla_i \nabla_j C(\vec{\sigma}) \Big|_{\substack{\vec{\rho} = (s_1 - s_2')\vec{x} \\ \vec{\sigma} = (s_2 - s_2')\vec{x}}}$$

$$\left. + \nabla_i \nabla_j C(\vec{\rho}) \nabla_i \nabla_j C(\sigma) \Big|_{\substack{\vec{\rho} = (s_1 - s_2')\vec{x} \\ \sigma = (s_2 - s_1')\vec{x}}} \right\} . \qquad (3.23)$$

Rather than attempting to evaluate these integrals precisely, we observe that $C(\vec{x})$ is roughly characterized by an overall scale, $[2k^2(\delta c/c)]^2$, and a correlation length L that describes the distance over which it falls to zero. Then each factor of ∇_i in the integrals may be replaced by 1/L and in estimating what remains we may regard $C(\vec{x})$ as being equal to $[2k^2(\delta c/c)]^2$ for $|\vec{x}| < L$ and zero otherwise. This gives the approximate results

$$\langle \Phi^2 \rangle = \frac{1}{L^4} \left(2k^2 \frac{\delta c}{c}\right)^2 \left(\frac{L}{R}\right) \qquad (3.24)$$

and

$$\langle \Omega^2 \rangle = \frac{1}{L^4} \left(2k^2 \frac{\delta c}{c}\right)^4 \left(\frac{L}{R}\right)^2 , \qquad (3.25)$$

the factors of L/R arising from the fact that $C\big((s_1 - s_2)R\big)$ will be large only for $(s_1 - s_2) \le L/R$.

We now can put the error estimates in a manageable form:

$$\frac{p_1}{p_0} \simeq \frac{R^2}{8k^2} \sqrt{\langle \Phi^2 \rangle} \simeq \frac{R^2}{8k^2} \left(2k^2 \frac{\delta c}{c}\right) \sqrt{\frac{L}{R}} \frac{1}{L^2} = \frac{1}{4} \frac{\delta c}{c} \left(\frac{R}{L}\right)^{3/2} , \qquad (3.26)$$

and

$$\frac{p_2}{p_1} \cong \frac{R}{3k} \frac{\sqrt{\langle \Omega^2 \rangle}}{\sqrt{\langle \phi^2 \rangle}} \cong \frac{R}{3k}\left(2k^2 \frac{\delta c}{c}\right)\sqrt{\frac{L}{R}} = \frac{\delta c}{c}\left(\frac{R}{L}\right)(kL) \; . \tag{3.27}$$

The most important feature of these ratios is their different R dependence: p_1/p_0 grows much more rapidly with R than does p_2/p_1. Consequently the approximation of neglecting terms like p_2 (the supereikonal approximation) should be good out to much larger values of R than the approximation of neglecting both p_1 and p_2 (the normal eikonal), provided that the coefficients in Eqs. (3.26) and (3.27) allow a region of R where $p_2/p_1 < 1$ while $p_1/p_0 \nless 1$. To illustrate the sort of distances involved, we choose $L = 10^2$m, $\lambda = 10$ m, $\delta c/c = 10^{-4}$ (these are roughly the parameter sizes we are interested in for deep ocean sound propagation). Thus

$$\frac{p_1}{p_0} < 1 \text{ for } R \lesssim 15 \text{ km}$$

and

$$\frac{p_2}{p_1} < 1 \text{ for } R \lesssim 3000 \text{ km} \; .$$

The different R dependences of the two error estimates have a dramatic effect on the distance for which the two approximations are valid. It seems clear that while the normal eikonal is probably inadequate for discussing most long-distance oceanic sound propagation problems, the supereikonal approximation should be much more useful.

C. Further Simplifications

The supereikonal thus constitutes an approximate expression in closed form for the pressure with a domain of validity that will exceed the domain of validity of geometrical optics on straight-line paths provided that $(kL)^2(\delta c/c)^{4/3} < 1$. While these expressions are relatively

straightforward, further simplifications are possible when the propagation distance is large and the wavelength is short, and we wish to describe these next.

First note that the integral over $d^3\vec{q}$ in Eq. (3.12) can be evaluated explicitly to yield

$$p(\vec{x}) = \frac{(-i\pi)^{1/2}}{8\pi^2} \int_0^\infty \frac{d\beta}{\beta^{3/2}}\, e^{i(\beta k + x^2/4\beta + \beta I(\beta,\vec{x}) + i\epsilon} \quad . \tag{3.28}$$

When k and x are very large, this integral may conveniently be evaluated using the method of stationary phase. Let us define

$$f(\beta) = \beta k^2 + x^2/4\beta + \beta I(\beta,\vec{x}) - 3/2 \log \beta \ . \tag{3.29}$$

Then we find

$$p(\vec{x}) \approx \frac{\sqrt{2}}{8\pi} \frac{e^{if(\beta_0)}}{\sqrt{f''(\beta_0)}} \tag{3.30}$$

where β_0 is the point at which f' vanishes:

$$f'(\beta_0) = 0 \ . \tag{3.31}$$

Next we note that, so long as $\delta c/c \ll 1$, β_0 is very nearly given by

$$\beta_0 = x/2k \ , \tag{3.32}$$

the point at which $\beta^2 + x^2/4\beta$ has its maximum. Therefore our result is

$$p(\vec{x}) = \frac{e^{ikx}}{4\pi x} \exp \frac{ix}{2k}\, I\left(\frac{x}{2k}, \vec{x}\right) \tag{3.33}$$

We could, of course, use this result, which is now only in terms of the one complicated integral defining the function I, as our final expression for the pressure. But we can do even better than this, as follows.

Let us recall that the supereikonal approximation makes its first error in second-order in V, and to first order our expression for the pressure must be exact. Thus, $p_1(\vec{x})$, defined by

$$p_1(\vec{x}) = \frac{(-i\pi)^{1/2}}{8\pi^2} \int_{\infty}^{\infty} \frac{d\beta}{\beta^{3/2}} e^{i(\beta k^2 + x^2/4\beta + i\epsilon)} \beta I(\beta, \vec{x}) \tag{3.34}$$

must be the correct first-order expression for the pressure. But this integral can also be evaluated by the method of stationary phase, since we are interested in large values of k and x. If we define

$$g(\beta) = \beta k^2 + x^2/4\beta - 3/2 \log \beta + \log \beta I(\beta, x) \tag{3.35}$$

then we can write, approximately,

$$p_1(\vec{x}) = \frac{\sqrt{2}}{8\pi} \frac{e^{ig(\beta_0)}}{\sqrt{g''(\beta_0)}} \tag{3.36}$$

where now $g'(\beta_0) = 0$. For small $\delta c/c$, we again have $\beta_0 = x/2k$; hence we find

$$p_1(\vec{x}) = \frac{e^{ikx}}{4\pi x} \left[\frac{x}{2k} I\left(\frac{x}{2k}, \vec{x} \right) \right] . \tag{3.37}$$

Combining this with Eq. (3.33) yields[3]

$$p(\vec{x}) = p_0(\vec{x})\, e^{p_1(\vec{x})/p_0(\vec{x})} \tag{3.38}$$

where

$$p_0(\vec{x}) = \frac{e^{ikx}}{4\pi x} \tag{3.39}$$

is the zeroth order (in V) pressure and

$$p_1(\vec{x}) = \int d^3y \frac{e^{ik|x-y|}}{4\pi|x-y|} V(\vec{y}) \frac{e^{iky}}{4\pi y} \tag{3.40}$$

is the first order (in V) pressure. Let us define

$$X(\vec{x}) = p_1(\vec{x})/p_0(\vec{x})$$

$$= \frac{1}{4\pi}\int d^3\vec{y}\ \frac{x}{|\vec{x-y}|y}\ e^{ik(|x-y|+y-x)}\ V(\vec{y})\ . \qquad (3.41)$$

Then

$$p(\vec{x}) = p_0(\vec{x})\ e^{X(\vec{x})}\ . \qquad (3.42)$$

It will be convenient in what follows to study the quantity $p(\vec{x})/\langle p(\vec{x})\rangle$. Since we assume the fluctuations in sound speed to be gaussian, we have

$$\langle p(x)\rangle = p_0(\vec{x})\ e^{1/2\langle X(x)\rangle^2}\ ; \qquad (3.43)$$

thus

$$\frac{p(x)}{\langle p(x)\rangle} = e^{X(\vec{x})-1/2\langle X(\vec{x})\rangle^2}\ . \qquad (3.44)$$

Over the propagation distances for which this approximation is valid, $\langle p(\vec{x})\rangle$ does not differ appreciably from $p_0(\vec{x})$. For longer distances, it does, and a quite accurate approximate expression for it is derived in Appendix B. We find

$$\langle p(\vec{x})\rangle = \frac{e^{i\sqrt{k^2-\Sigma(k)x}}}{4\pi x} \qquad (3.45)$$

where

$$\Sigma(k) = \int \frac{d^3\vec{q}}{(2\pi)^3}\ \frac{\tilde{C}(\vec{k-q})}{k^2-q^2+i\epsilon} \qquad (3.46)$$

and $\tilde{C}(q)$ is the Fourier transform of the correlation function:

$$\tilde{C}(\vec{q}) = \int d^3\vec{x}\ e^{-i\vec{q}\cdot\vec{x}}\ C(\vec{x}) \qquad (3.47)$$

and

$$C(\vec{x}-\vec{y}) = \langle V(\vec{x})V(\vec{y})\rangle \ . \tag{3.48}$$

From Eq. (3.44), various statistical averages of the received pressure are readily derived. For example,

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = e^{\langle X(\vec{x}_1)X(\vec{x}_2)\rangle} \tag{3.49}$$

and

$$\frac{\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle^*} = e^{\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle} \ . \tag{3.50}$$

Our problem is thus reduced to studying averages of the function $X(\vec{x})$, and this we shall do in the following section.

IV APPLICATIONS OF THE SUPEREIKONAL FORMULA

Implementation of the simplified form of the supereikonal formula, derived in the previous section, requires the evaluation of the integral denoted $X(\vec{x})$, or what is perhaps more relevant, the evaluation of statistical averages of the integral such as $\langle X(\vec{x}_1)X(\vec{x}_2)\rangle$ and $\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle$. Before attempting this in general, however, it is of some interest to see how the conventional geometrical optics limit is obtained from this formula, so let us indulge in this digression first.

We recall that

$$X(\vec{x}) = \frac{1}{4\pi}\int d^3\vec{y}\, \frac{|\vec{x}|}{|\vec{x}-\vec{y}|\,|\vec{y}|}\, e^{ik(|\vec{x}-\vec{y}|+|\vec{y}|-|\vec{x}|)}\, V(\vec{y}) \quad . \tag{4.1}$$

Let us choose the point of observation $\vec{x}$ to lie on the z axis, so that $\vec{x} = (0,0,R)$. For very short wavelengths--that is, in the limit $k \to \infty$-- the only important part of the region of integration in Eq. (4.1) is near the z axis. Hence we may expand $V(\vec{y})$ around the z axis:

$$V(x,y,z) \approx V(0,0,z) + x\left.\frac{\partial V}{\partial x}\right|_{x=y=0} + y\left.\frac{\partial V}{\partial y}\right|_{x=y=0} + \frac{1}{2}x^2\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=y=0} + \frac{1}{2}y^2\left.\frac{\partial^2 V}{\partial y^2}\right|_{x=y=0} + xy\left.\frac{\partial^2 V}{\partial x\,\partial y}\right|_{x=y=0} + \ldots \tag{4.2}$$

We further note that

$$|\vec{y}| \approx z + \frac{x^2+y^2}{2z} + \ldots \tag{4.3}$$

and

$$|\vec{x}-\vec{y}| \approx R-z + \frac{x^2+y^2}{2(R-z)} + \ldots ; \tag{4.4}$$

finally, the z integration can be restricted to the range $z = 0$ to $z = R$. With these approximations, the integral in Eq. (4.1) can be readily evaluated. We note that only terms even in x and y survive; thus the terms in $\partial V/\partial x$, $\partial V/\partial y$, and $\partial^2 V/\partial x \partial y$ in Eq. (4.2) may be dropped, and we obtain

$$X(\vec{x}) = \frac{R}{4\pi} \int_0^R \frac{dz}{z(R-z)} \int_\infty^\infty dx \int_\infty^\infty dy \; \exp \frac{ik}{2}\left(\frac{1}{z} + \frac{1}{R-z}\right)(x^2 + y^2)$$

$$\left\{ V\Big|_{x=y=0} + \frac{1}{2}\left(x^2 \frac{\partial^2 V}{\partial x^2}\Bigg|_{x=y=0} + y^2 \frac{\partial^2 V}{\partial y^2}\Bigg|_{x=y=0} \right)\right\} . \qquad (4.5)$$

Carrying out the dx and dy integrals yields

$$X(\vec{x}) = -\frac{i}{2k}\int_0^R V\Big|_{x=y=0} dz + \frac{1}{4k^2}\int_0^R \nabla^2 V\Big|_{x=y=0} (z - z^2/R)\, dz \qquad (4.6)$$

which is recognizable as the usual formula of geometrical optics on straight-line paths, for both phase and amplitude of the received signal at the point $\vec{x}$.

Let us now return to the more general situation, and attempt to compute the statistical average $\langle X(\vec{x}_1) X^*(\vec{x}_2) \rangle$. We have

$$\langle X(\vec{x}_1) X(\vec{x}_2)^* \rangle = \left(\frac{1}{4\pi}\right)^2 \int d^3\vec{y}_1 \int d^3\vec{y}_2 \; C(y_1 - y_2)$$

$$\frac{|\vec{x}_1|}{|x_1 - y_1|\,|\vec{y}_1|} \cdot \frac{|\vec{x}_2|}{|x_2 - y_2|\,|\vec{y}_2|} \, e^{ik\left(|x_1 - y_1| + |\vec{y}_1| - |\vec{x}_1|\right)} \, e^{-ik\left(|x_2 - y_2| + |\vec{y}_2| - |\vec{x}_2|\right)} .$$

(4.7)

We are interested in the situation where the correlation length L as well as the wavelength are both small compared to $|\vec{x}_1|$ and $|\vec{x}_2|$; thus, $|y_1 - y_2| \ll |\vec{x}_{1,2}|$ and $|x_1 - x_2| \ll |\vec{x}_{1,2}|$. We are therefore invited to change to "center of mass" and "relative" coordinates in Eq. (4.7) and to assume

that the relative coordinates are small compared to the center-of-mass coordinates. Thus we define

$$\vec{R} = \frac{\vec{x}_1 + \vec{x}_2}{2}, \quad \vec{x} = \vec{x}_1 - \vec{x}_2$$

$$\vec{R}' = \frac{\vec{y}_1 + \vec{y}_2}{2}, \quad \vec{y} = \vec{y}_1 - \vec{y}_2 \tag{4.8}$$

and we write, approximately, when $|\vec{R}| >> |\vec{x}|$,

$$|\vec{x}_1| = |\vec{R} + \frac{\vec{x}}{2}| \approx R + \frac{1}{2}\hat{R}\cdot\vec{x} \tag{4.9}$$

with similar expressions for $|\vec{x}_2|$, $|\vec{y}_1|$, $|\vec{y}_2|$, $|x_1 \vec{-} y_1|$ and $|x_2 \vec{-} y_2|$. Then Eq. (4.7) becomes

$$\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle \approx \left(\frac{1}{4\pi}\right)^2 \int d^3\vec{y} \int d^3\vec{R}' C(\vec{y})$$

$$\left(\frac{R}{R'R''}\right)^2 e^{ik[\hat{R}''(x\vec{-}y)+\hat{R}'\cdot\vec{y}-\hat{R}\cdot\vec{x}]} \tag{4.10}$$

where we define $\vec{R}'' = \vec{R} - \vec{R}'$. We define the Fourier transform of the correlation function by

$$\tilde{C}(\vec{q}) = \int d^3 q\, \vec{e}^{-i\vec{q}\cdot\vec{x}} C(\vec{x}) ; \tag{4.11}$$

Then

$$\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle = \left(\frac{1}{4\pi}\right)^2 \int d^3\vec{R}' \left(\frac{R}{R'R''}\right)^2$$

$$e^{ik(\hat{R}''-\hat{R})\cdot\vec{x}}\, \tilde{C}\left(k(\hat{R}'' - \hat{R}')\right) . \tag{4.12}$$

At this point let us observe as an aside that a formula analogous to Eq. (4.12) can evidently be written for the average $\langle X(\vec{x}_1)X(\vec{x}_2)\rangle$:

$$\langle X(\vec{x}_1)X(\vec{x}_2)\rangle = \left(\frac{1}{4\pi}\right)^2 \int d^3\vec{R}' \left(\frac{R}{R'R''}\right)^2$$

$$e^{2ik(R'+R''-R)}\, \tilde{C}(0) . \tag{4.13}$$

This formula, we note, involves only the correlation function at zero argument, and does not depend on the separation $\vec{x} = \vec{x}_1 - \vec{x}_2$ between the two observation points. Evidently the situation here is much simpler than that which obtains in the calculation of $\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle$. The integral in Eq. (4.13) can in fact be evaluated analytically. We first note that

$$d^3R = \frac{2\pi}{R} R'dR' \; R''dR'' \tag{4.14}$$

and hence we find

$$\langle X(\vec{x}_1)X(\vec{x}_2)\rangle = \frac{R\tilde{C}(0)}{4}\int_{-R/2}^{R/2} dx \int_{R/2}^{\infty} dy \; \frac{e^{4ik(y-R/2)}}{y^2-x^2} \tag{4.15}$$

where we have defined new variables of integration through

$$x = \frac{R'-R''}{2} \quad \text{and} \quad y = \frac{R'+R''}{2} \; . \tag{4.16}$$

Finally, when $kR \gg 1$, Eq. (4.15) becomes

$$\langle X(\vec{x}_1)X(\vec{x}_2)\rangle = \frac{i\tilde{C}(0)}{8\pi k}(C + \log 4kR - i\pi/2) \tag{4.17}$$

where C = 0.577... is Euler's constant. The corresponding pressure average then becomes

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = e^{[i\tilde{C}(0)/8\pi k](C+\log 4kR-i\pi/2)}$$

and, we recall,

$$\tilde{C}(0) = \int d^3\vec{x}\; C(\vec{x}) \; .$$

There is, then, a distance-independent amplitude exp $\tilde{C}(0)/16k$ that is very small, and of no interest, and a small phase $\tilde{C}(0)/8\pi k\ (C + \log 4kR)$ growing logarithmically with distance, of almost no interest. We recall that the entire approximation is valid only out to a distance

$R \sim L(1/kL)^2(c/\delta c)^2$, so the growth in phase in limited to log $[4/kL(c/\delta c)^2]$; furthermore, as we remarked earlier, there is no dependence on $\vec{x} = \vec{x}_1 - \vec{x}_2$.

Let us now return to Eq. (4.12) and the more interesting--and also more difficult--average $\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle$.

To begin with, we shall make a rough estimate of the integral in Eq. (4.12). First, suppose the correlation function $C(\vec{x})$ is spherically symmetric; then $\tilde{C}(\vec{q})$ is also spherically symmetric, and let us suppose it cuts off for $q^2 > 1/L^2$, where L is the correlation length. The allowed region of integration in Eq. (4.12) is then

$$k^2(2 - 2\hat{R}'\hat{R}'') \lesssim 1/L^2$$

or, if θ is the angle between $\hat{R}'$ and $\hat{R}''$,

$$\theta \lesssim 1/kL \ .$$

If $kL \gg 1$, this is a small angle. The integration volume is therefore a sausage-shaped region of length R and maximum radius of order R/kL, with the origin and the point R at opposite ends of the sausage.

Next we must estimate the geometrical factor $(R/R'R'')^2$. This, averaged over the region of integration, turns out to be of the order $1/R^2$, as one might expect on dimensional grounds alone. On the basis of these comments, let us first estimate $\langle|X(\vec{R})|^2\rangle$; that is, let us take $x = 0$. Then, roughly, Eq. (4.12) implies

$$\langle|X(\vec{R})|^2\rangle \sim \frac{1}{R^2} R\cdot\left(\frac{R}{kL}\right)^2$$

$$\sim \frac{R\tilde{C}}{(kL)^2} \ .$$

Since $C \sim k^4L^3(\delta c/c)^2$, we find

$$\langle|X(\vec{R})|^2\rangle \sim k^2RL\left(\frac{\delta c}{c}\right)^2 \ .$$

Next, if we expand the exponential in Eq. (4.12) in powers of $\vec{x}$, we see that the linear term in $\vec{x}$ vanishes by symmetry, and the quadratic term is

$$\left(\frac{1}{4\pi}\right)^2 \int d^3R' \left(\frac{R}{R'R''}\right)^2 [k(\hat{R}''-\hat{R})\cdot\vec{x}]^2 \tilde{C}[k(\hat{R}''-\hat{R}')] \ .$$

Now inside the sausage-shaped column, $\hat{R}'-\hat{R}$ is nearly perpendicular to $\hat{R}$, and is very small, of order $1/kL$ in length. Thus, only the component of $\vec{x}$ perpendicular to $\hat{R}$ survives, and we may estimate

$$[k(\hat{R}''-\hat{R})\cdot\vec{x}]^2 \sim (x_\perp/L)^2 \ .$$

As a result, our crude estimate gives

$$\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle \sim k^2RL\left(\frac{\delta c}{c}\right)^2[1-(x_\perp/L)^2$$

and hence

$$\frac{\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle^*} \sim \exp\left\{k^2RL\left(\frac{\delta c}{c}\right)^2[1-(x_\perp/L)^2]\right\} \ .$$

Thus $\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle$ vanishes unless $k^2RL(x_\perp/L)^2(\delta c/c)^2 < 1$; hence the transverse coherence length is

$$D \sim \lambda \sqrt{\frac{L}{R}}\left(\frac{c}{\delta c}\right) \ .$$

The longitudinal (i.e., parallel to $\hat{R}$) coherence length is, in contrast, infinite.

We may also estimate $\langle X(x_1)X^*(x_2)\rangle$ in cylindrical rather than spherical symmetry. Let us suppose that $\tilde{C}(\vec{q})$ is cylindrically symmetric in a horizontal plane, and cuts off when $q_H > 1/L_H$, and $q_V > 1/L_V$, where L_H and L_V are horizontal and vertical correlation lengths, respectively. Let us consider the case that $\vec{R}$ lies in a horizontal plane. Then the region of integration in Eq. (4.12) is a flattened sausage of length R, vertical height R/kL_V, width R/kL_H, and consequently of volume $R^3/k^2L_HL_V$.

The factor $(R/R'R'')^2$ still averages to $1/R^2$. Then

$$\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle \sim \frac{R\tilde{C}}{k^2 L_H L_V}\left[1 - \left(\frac{x_V}{L_V}\right)^2 - \left(\frac{x_H}{L_H}\right)^2\right]$$

where x_V and x_H are vertical and horizontal components of $\vec{x}$ (the longitudinal component of $\vec{x}$ does not enter, as before). Next, since we now have

$$\tilde{C} \sim k^4 \left(\frac{\delta c}{c}\right)^2 L_H^2 L_V \quad ,$$

we find

$$\frac{\langle p(\vec{x}_1)p^*(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle^*} \sim \exp\left\{k^2 R\, L_H \left(\frac{\delta c}{c}\right)^2 \left[1 - \left(\frac{x_V}{L_V}\right)^2 - \left(\frac{x_H}{L_H}\right)^2\right]\right\};$$

hence the vertical and horizontal correlation lengths are

$$D_V \sim L_V \sqrt{\frac{\lambda^2}{RL_H}} \left(\frac{c}{\delta c}\right)$$

and

$$D_H \sim L_H \sqrt{\frac{\lambda^2}{RL_H}} \left(\frac{c}{\delta c}\right) .$$

Obviously $D_V/D_H \sim L_V/L_H$, as is only to be expected.

These crude estimates are, perhaps, useful, but it would obviously be desirable to be able to do better. Let us therefore again return to Eq. (4.12) and specialize to the case $x = 0$. Thus

$$\langle |X(\vec{R})|^2\rangle = \left(\frac{1}{4\pi}\right)^2 \int d^3\vec{R}' \left(\frac{R}{R'R''}\right)^2 \tilde{C}\left(k(\hat{R}''-\hat{R}')\right) . \qquad (4.18)$$

Let us suppose we have spherical symmetry. Then changes of variables just like those used in obtaining Eq. (4.15) yield the expression

$$\langle |X(\vec{R})|^2\rangle = \frac{R}{4\pi}\int_{-R/2}^{R/2} dx \int_{R/2}^{\infty} dy \, \frac{1}{y^2-x^2} \; \tilde{C}\left(2k^2 \, \frac{y^2-R^2/4}{y^2-x^2}\right) \qquad (4.19)$$

where we have, because of the spherical symmetry, replaced $\tilde{C}(\vec{q})$ by $\tilde{C}(q^2)$. The distance R can be scaled out of Eq. (4.19) to obtain

$$\langle |X(\vec{R})|^2 \rangle = \frac{R}{4\pi} \int_{-1}^{1} dx \int_{1}^{\infty} dy \; \frac{1}{y^2 - x^2} \, \tilde{C} \left(2k^2 \cdot \frac{y^2 - 1}{y^2 - x^2} \right) \quad ; \tag{4.20}$$

thus $\langle |X(\vec{R})|^2 \rangle$ is simply linear in R.* A further change of variables finally results in

$$\langle |X(\vec{R})|^2 \rangle = \frac{R}{4\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \left(\sin^{-1} \sqrt{z} \right) \tilde{C}(2k^2 z) \; . \tag{4.21}$$

In this form we may evidently easily compute $\langle |X|^2 \rangle$, and thereby $\langle |p|^2 \rangle$, numerically for any given correlation function $\tilde{C}$. Further analytic progress, however, is possible only for sufficiently simple correlation functions.

As an illustration, suppose we look at the example of an exponential correlation function:

$$C(\vec{x}) = C \, e^{-x/L} \tag{4.22}$$

Then we find

$$\tilde{C}(q^2) = 8\pi C L^3 \left(\frac{1}{1 + q^2 L^2} \right)^2 \; . \tag{4.23}$$

---

*This statement is, of course, only true when R is not too large. The integrand in Eq. (4.19) should really be modified by a factor

$$\exp - 4\alpha(y - R/2)$$

where $\alpha = \mathrm{Im}\, \Sigma(k)/2k$ is the damping produced by the "self-energy bubbles" on the propagator as described in Appendix B. Thus the integrand in Eq. (4.20) contains a factor

$$\exp - 2\alpha R(y-1) \; ;$$

hence, if $\alpha R \gtrsim 1$, the linear dependence on R becomes modified. However, when $\alpha R \gtrsim 1$, R is so large that the supereikonal approximation fails, and the formalism we are using is not applicable anyway. Over the range of validity of what we are doing, then, it is valid to neglect this factor, and $\langle |X(\vec{R})|^2 \rangle$ is indeed linear in R.

With this choice of $\tilde{C}$, and when kL is large, the dominant contribution to the integral in Eq. (4.21) comes for small z; hence we may approximate $\sin^{-1}\sqrt{z}$ in Eq. (4.21) by $\sqrt{z}$. The integral may then be evaluated analytically, and for kL large we find

$$\langle |X(\vec{R})|^2 \rangle = \frac{RCL}{k^2} \quad . \tag{4.24}$$

Setting $C = 4k^4(\delta c/c)^2$ yields, finally

$$\langle |X(\vec{R})|^2 \rangle = 4k^2 RL \left(\frac{\delta c}{c}\right)^2 \quad . \tag{4.25}$$

This may be compared with the very crude estimate made earlier; we see that this more accurate calculation, for the exponential correlation function, differs from that estimate by a factor of 4, which is, we note, not totally insignificant numerically. To complete this example, we deduce that the mean square pressure is

$$\frac{\langle |p(x)|^2 \rangle}{|\langle p(x) \rangle|^2} = e^{[4(\delta c/c)^2 k^2 L]R}$$

and this, we recall, is valid out to distances $R \sim L(1/kL)^2(c/\delta c)^2$.

To conclude, let us summarize what we have found:

(1) For an arbitrary correlation function C(x),

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = \exp \frac{\tilde{C}(0)}{16k} \exp \frac{i\tilde{C}(0)}{8\pi k} (C + \log 4kR)$$

where $\tilde{C}(0) = \int d^3\vec{x}\, C(\vec{x})$, where C is Euler's constant, and where $R = |\vec{x}_1 + \vec{x}_2/2|$.

(2) For an exponential correlation function $C(\vec{x}) = C\, e^{-x/L}$,

$$\frac{\langle |p(\vec{R})|^2 \rangle}{|\langle p(\vec{R}) \rangle|^2} = e^{(CL/k^2)R} ,$$

where $R = |\vec{x}|$.

(3) For an arbitrary spherically symmetric correlation function $C(\vec{x})$, of correlation length L,

$$\frac{\langle p(\vec{x}_1) p^*(\vec{x}_2) \rangle}{\langle p(\vec{x}_1) \rangle \langle p(\vec{x}_2) \rangle^*} \sim e^{[\tilde{C}(0)/(kL)^2][1-(x_\perp/L)^2 R]}$$

where $\vec{x} = \vec{x}_1 - \vec{x}_2$, and $x_\perp$ is the component perpendicular to $\vec{x}_1 + \vec{x}_2/2 = \vec{R}$. Thus, the transverse coherence length is of the order

$$D \sim L \cdot \frac{kL}{\sqrt{\tilde{C}(0)}} .$$

Appendix A

GEOMETRIC OPTICS WITH STRAIGHT-LINE PROPAGATION

Appendix A

GEOMETRIC OPTICS WITH STRAIGHT-LINE PROPAGATION

We wish to briefly review the conditions under which geometrical optics with straight-line paths are valid. This is also the same as the result obtained from the supereikonal formula in the very-short-wavelength limit, as outlined in Eqs. (4.2) through (4.6). The first term in Eq. (4.6) is what is obtained if all quadratic terms in the momenta are neglected in the derivation of the supereikonal formula; this is what is known as the conventional eikonal approximation.

The wave equation,

$$(\nabla^2 + k^2)p(x) = V(x)p(x) \tag{A-1}$$

may be rewritten as the two equations

$$-2k\,\hat{x}\cdot\vec{\nabla}\delta\chi - (\vec{\nabla}\delta\chi)^2 - \frac{2\hat{x}}{x}\cdot\vec{\nabla}\log A/A_0 + \nabla^2\log A/A_0 + (\vec{\nabla}\log A/A_0)^2 = V \tag{A-2}$$

and

$$\nabla^2\delta\chi - 2\,\frac{\hat{x}}{x}\cdot\vec{\nabla}\delta\chi + 2k\,\hat{x}\cdot\vec{\nabla}\log A/A_0 + 2V\delta\chi\cdot\vec{\nabla}\log A/A_0 = 0 \tag{A-3}$$

through the definition

$$-i\log p(x) - kx + \delta\chi(x) + i\log 4\pi x - i\log A/A_0 \tag{A-4}$$

where $A_0 \equiv 1/4\pi x$, and where $\delta\chi$ and A are real. Equation (A-4) may also be written

$$p(x) = \frac{e^{ikx}}{4\pi x} \cdot (A/A_0)\, e^{i\delta\chi} . \tag{A-5}$$

Thus, $\delta\chi$ is the change in phase and A/A_0 is the change in amplitude, due to the presence of the potential V(x).

Now in Eq. (A-3) let us assume

(a) $(\vec{\nabla}\delta\chi)^2 << V$

(b) $\nabla^2 \log A/A_0 << V$

(c) $\frac{\hat{x}}{x} \cdot \vec{\nabla} \log A/A << V$

(d) $(\vec{\nabla} \log A/A_0)^2 << V$

and in Eq. (A-4) let us assume

(e) $\vec{\nabla}\delta\chi \cdot \vec{\nabla} \log A/A_0 << \nabla^2\delta\chi,\ \frac{\hat{x}}{x} \cdot \vec{\nabla}\delta\chi,$
$kx \cdot \vec{\nabla} \log A/A_0$.

Then Eqs. (A-3) and (A-4) reduce to

$$-2k\ x \cdot \vec{\nabla}\delta\chi = V \tag{A-6}$$

and

$$\nabla^2\delta\chi - 2\frac{\hat{x}}{x} \cdot \nabla\delta\chi + 2k\ x \cdot \nabla \log A/A_0 = 0 \tag{A-7}$$

which are the equations of geometrical optics with straight-line paths, or of the conventional eikonal approximation. Their solution is trivially seen to be

$$\delta\chi(r,\theta,\phi) = -\frac{1}{2k}\int_0^r dr'\ V(r',\theta,\phi) \tag{A-8}$$

and

$$\log\frac{A(r,\theta,\phi)}{A_0(r,\theta,\phi)} = \frac{1}{4k^2}\int_0^r dr'\left(r' - \frac{r'^2}{r}\right)\nabla^2 V(R',\theta,\phi) . \tag{A-9}$$

Given these solutions, we can now go back and check under what conditions Assumptions a through e were valid.

First we note, from Eq. (A-8), that

$$\frac{\partial}{\partial r}\,\delta\chi \sim \frac{V}{k}$$

but that

$$\nabla_{\perp}\,\delta\chi \sim \frac{R}{L}\,\frac{V}{k}$$

where $\nabla_{\perp}$ means the gradient transverse to the path of integration. Hence,

$$\nabla^2\delta\chi \sim \frac{R}{L^2}\,\frac{V}{k}\ .$$

Similarly, we see from (A-9) that

$$\frac{\delta}{\delta r}\log A/A_0 \sim \frac{R}{L^2}\,\frac{V}{k^2}$$

but that

$$\nabla_{\perp}\log A/A_0 \sim \frac{R^2}{L^3}\,\frac{V}{k^2}$$

and that

$$\nabla^2\log A/A_0 \sim \frac{R^2}{L^4}\,\frac{V}{k^2}\quad .$$

Using these estimates, our assumptions a through e now require that:

$$\text{(a)}\quad \left(\frac{R}{L}\,\frac{V}{k}\right)^2 << V$$

$$\text{(b)}\quad \frac{R^2}{L^4}\,\frac{V}{k^2} << V$$

$$\text{(c)}\quad \frac{1}{R}\cdot\frac{R}{L^2}\,\frac{V}{k^2} << V$$

$$\text{(d)}\quad \left(\frac{R^2}{L^3}\,\frac{V}{k^2}\right)^3 << V$$

$$\text{(e)}\quad \frac{R}{L}\,\frac{V}{k}\cdot\frac{R^2}{L^3}\,\frac{V}{k^2} << \frac{V}{k}\,\frac{R}{L^2}\quad .$$

Among these conditions we note that a and b imply d; that a implies e and that c simply requires $kL \ll 1$. Thus only the two conditions a and b count, and they require

$$(a') \quad R \ll (kL)\sqrt{1/V}$$

$$(b') \quad R \ll L(kL) \;.$$

These yield very limited ranges of validity indeed; however, we may improve them by recalling that we are really only interested in statistical fluctuations in the pressure. Thus we are justified in multiplying the left-hand sides of Conditions a and b by $1/\sqrt{N}$, where $N \sim R/L$ is the number of traversed inhomogeneities. Then a and b become

$$(a'') \quad \sqrt{\frac{L}{R}}\left(\frac{R}{L}\,\frac{V}{k}\right)^2 \ll V$$

$$(b'') \quad \sqrt{\frac{L}{R}}\left(\frac{R^2}{L^4}\,\frac{V}{k^2}\right) \ll V$$

which yield the limits

$$(a''') \quad R \ll L(k^2/V)^{2/3}$$

and

$$(b''') \quad R \ll L(kL)^{4/3}$$

respectively.

To conclude, let us recall that $V = 2k^2\,\delta c/c$; thus we have to have

$$(a''') \quad R \ll L(c/\delta c)^{2/3}$$

and

$$(b''') \quad R \ll L(2\pi L/\lambda)^{4/3}$$

in order to be able to use geometrical optics, or the eikonal approximation, or whatever one wants to call it.

Condition a‴, we note, was also obtained in our derivation of the supereikonal approximation, as the condition for the validity of the ordinary eikonal approximation.

Finally, for our standard values, the validity conditions of geometrical optics state that we must have

$$(a''') \quad R \ll 10^2 \cdot (10^4)^{2/3} m \sim 50 \text{ km}$$

$$(b''') \quad R \ll 10^2 \cdot (60)^{4/3} m \sim 25 \text{ km}.$$

Appendix B

CALCULATICN OF SOUND PRESSURE AT GREAT DISTANCES FROM A SOURCE

Appendix B

CALCULATION OF SOUND PRESSURE AT GREAT DISTANCES FROM A SOURCE

We wish here to briefly outline a technique different from that discussed in the text for handling calculations of the sound pressure at very great distances from a source. This technique permits a very good calculation of $\langle p(\vec{x}) \rangle$, but is not well suited to deriving useful analytic results for more complicated averages such as $\langle |p(\vec{x})|^2 \rangle$.

The idea is based on the use of the methods of quantum field theory, and it has appeared, albeit in what seems to be unnecessarily complicated language, in the literature.[4]

We start with the perturbation series for the pressure, Eq. (3.4) of the text:

$$p(\vec{x}) = p_0(\vec{x}) + \int d^3\vec{x}_1 \, G_0(\vec{x}-\vec{x}_1)V(\vec{x}_1)p_0(\vec{x}_1)$$
$$+ \int d^3\vec{x}_1 \int d^3\vec{x}_2 G_0(\vec{x}-\vec{x}_1)V(\vec{x}_1)G_0(\vec{x}_1-\vec{x}_2)V(\vec{x}_2)p_0(\vec{x}_2)$$
$$+ \dots \tag{B-1}$$

and we note that, for unit source strength,

$$p_0(\vec{x}) = G_0(\vec{x}) \ . \tag{B-2}$$

Let us calculate the average pressure from Eq. (B-1). Recall that $\langle V(x) \rangle = 0$, and let us assume that only two point correlations matter, so that

$$\langle V(\vec{x}_1) \dots V(\vec{x}_n) \rangle = 0 \quad \text{if } n \text{ is odd}$$

while

$$\langle V(\vec{x}_1)\ldots V(\vec{x}_n)\rangle = \sum_{\text{permutations}} C(\vec{x}_1-\vec{x}_2)\ldots C(\vec{x}_{n-1}-\vec{x}_n)$$

if n is even.

Thus we find

$$\langle p(\vec{x})\rangle = G_0(\vec{x}) + \int d^3\vec{x}_1 \int d^3\vec{x}_2\; G_0(\vec{x}-\vec{x}_1)G_0(\vec{x}_1-\vec{x}_2) C(\vec{x}-\vec{x}_2)G_0(\vec{x}_2) + \ldots \qquad \text{(B-3)}$$

This is obviously precisely the perturbation expansion of the propagator for a particle whose free propagator is given by $G_0(x)$, which can emit and absorb another particle whose propagator is $C(x)$. The set of Feynman diagrams comprising $\langle p(\vec{x})\rangle$ is shown in Figure B-1, and the Feynman rules for calculating the contribution of any diagram are evident from Eq. (B-3).

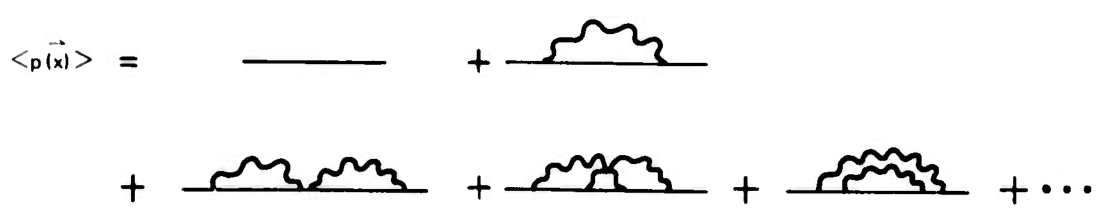

FIGURE B-1 FEYNMAN DIAGRAMS FOR THE "PROPAGATOR" $\langle p(\vec{x})\rangle$. The straight line represents the propagator G_0; the wavy line represents the propagator C.

Similar expressions can be easily derived from Eq. (B-1) for other averages of interest. For example, the quantity $\langle p(\vec{x}_1)p(\vec{x}_2)\rangle$ is represented by the diagrams shown in Figure B-2, and is evidently a vertex function, in which two "particles" emitted at the origin propagate to the points $\vec{x}_1$ and $\vec{x}_2$ emitting and reabsorbing the "C type" particles as they go. The quantity $\langle p(\vec{x}_1)p(\vec{x}_2)^*\rangle$ is represented by the same set of Feynman graphs, except that one of the solid lines now stands for propagators G_0^* rather than G_0.

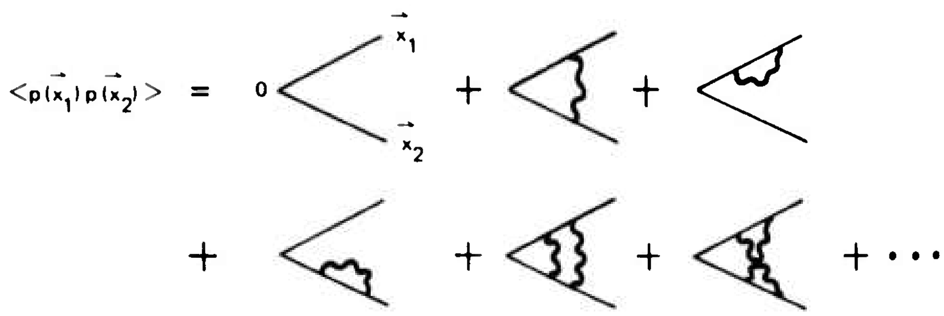

FIGURE B-2 FEYNMAN DIAGRAMS FOR THE "VERTEX FUNCTION" $\langle p(\vec{x}_1)p(\vec{x}_2)\rangle$

Now let us return to Eq. (B-3). Since $\delta c/c$ is so small, the quantity C is very small. Hence the dominant set of graphs contributing to $\langle p(\vec{x})\rangle$ will be those shown in Figure B-3. These are trivially summed if we go to momentum space:

$$\langle p(\vec{x})\rangle = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \left\{ \frac{1}{q^2-k^2+i\epsilon} + \frac{1}{q^2-k^2+i\epsilon} \int \frac{d^3\vec{q}\,'}{(2\pi)^3} \frac{\tilde{C}(\vec{q}-\vec{q}\,')}{q'^2-k^2+i\epsilon} + \cdots \right\}$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{1}{q^2-k^2+\Sigma(\vec{q})+i\epsilon} , \qquad \text{(B-4)}$$

where

$$\Sigma(\vec{q}) = \int \frac{d^3\vec{q}\,'}{(2\pi)^3} \frac{\tilde{C}(\vec{q}-\vec{q}\,')}{k^2-q'^2} . \qquad \text{(B-5)}$$

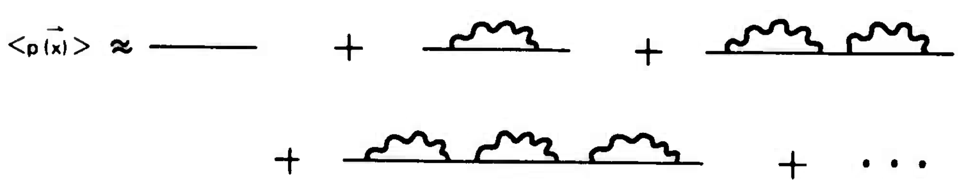

FIGURE B-3 DOMINANT FEYNMAN DIAGRAMS CONTRIBUTING TO $\langle p(\vec{x})\rangle$ WHEN $\delta c/c$ IS SMALL

At very large distances $R = |\vec{x}|$, the leading term in $\langle p(\vec{x})\rangle$ will come from the singularity in $\vec{q}$ in the Fourier transform; this occurs when

$$q^2 - k^2 + \Sigma(\vec{q}) = 0 \tag{B-6}$$

which, when Σ is small, is approximately given by

$$q^2 = k^2 - \Sigma(k) \ . \tag{B-7}$$

Thus we arrive at the answer:

$$\langle p(x)\rangle = \frac{e^{i\sqrt{k^2-\Sigma(k)}R}}{4\pi R}$$

$$\simeq \frac{e^{ikR}}{4\pi R} \cdot e^{-[i\Sigma(k)/2k]R} \ . \tag{B-8}$$

In view of the $i\epsilon$ in the denominator of the integrand in Eq. (B-5), $\Sigma(k)$ has an imaginary part as well as a real part. This gives rise to a damping of $\langle p(\vec{x})\rangle$. If we forget the uninteresting small phase charge produced by Re $\Sigma(k)$, we can write

$$\langle p(\vec{x})\rangle = \frac{e^{ikR}}{4\pi R}\, e^{-\alpha R} \tag{B-9}$$

where

$$\alpha = -\mathrm{Im}\,\Sigma(k) = \Pi \int \frac{d^3q}{(2\pi)^3}\, \tilde{C}(k-\vec{q}\,')\,\delta(q'^2-k^2) \ . \tag{B-10}$$

For example, if we choose

$$C(\vec{x}) = C\, e^{-|\vec{x}|/L} \tag{B-11}$$

we find, from Eq. (B-10),

$$\alpha = CL/4k^2. \tag{B-12}$$

The use of the Feynman diagram technique is easy and valid for calculating $\langle p(\vec{x})\rangle$. It turns out that it is also easy to calculate $\dfrac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle}$; the lowest-order diagram shown in Figure B-4 is all that

$$\frac{\langle p(\vec{x}_1)\,p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} \approx$$

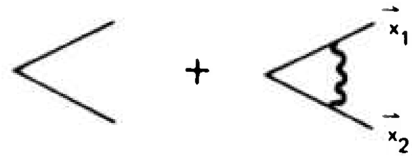

FIGURE B-4 THE ONLY DIAGRAMS NEEDED TO CALCULATE $\langle p(\vec{x}_1)p(\vec{x}_2)\rangle/\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle$ WHEN $\delta c/c$ IS SMALL

it is necessary to include. The result obtained by doing this is

$$\frac{\langle p(\vec{x}_1)p(\vec{x}_2)\rangle}{\langle p(\vec{x}_1)\rangle\langle p(\vec{x}_2)\rangle} = 1 + \langle X(\vec{x}_1)X(\vec{x}_2)\rangle \tag{B-13}$$

where $X(\vec{x})$ is as defined in Eq. (3.41). Thus, the Feynman diagram technique, for this quantity, coincides exactly with what we obtain from the supereikonal. (We recall that the expansion $\exp\langle X(\vec{x}_1)X(\vec{x}_2)\rangle \approx 1 + \langle X(\vec{x}_1)X(\vec{x}_2)\rangle$ is quite valid for this case.)

It is, however, not easy to calculate $\langle p(\vec{x}_1)p(\vec{x}_2)^*\rangle$ by this method. The first Feynman diagram coincides with the first term in $\langle X(\vec{x}_1)X^*(\vec{x}_2)\rangle$ of the supereikonal, but here it is <u>not</u> valid to stop after one term, even when $\delta c/c$ is small. An infinite series of diagrams is needed, and (a part) of such a series is what is provided by the supereikonal formula.

REFERENCES

1. R. M. Kennedy, "Phase and Amplitude Fluctuations in Propagating Through a Layered Ocean," J. Acoust. Soc. Am., Vol. 46, p. 737 (1969).

2. W. Munk and M. Rosenbluth, "Scintillations of Sound in the Sea," preprint, Scripps Institute of Oceanography, La Jolla, California (1973).

3. This result is identical to that obtained through an approximation scheme known as "Rytov's method" (see L. Chernov, Wave Propagation in a Random Medium, McGraw-Hill Book Co., New York, N.Y., 1960). We are grateful to Dr. J. Sanborn for calling this fact to our attention.

4. V. I. Tatarski, Wave Propagation in a Turbulent Media, 3rd Ed., Part V (McGraw-Hill Book Co., Inc., 1961).